

Moduli spaces of maps with two critical points

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Abstract

We give directly a defining equation of the symmetry locus, a singular part of the moduli space of the quadratic rational maps. We show a characterization of this locus. We can expand analogous discussion for the cubic polynomials and give a “chart” making a comparison between properties of these moduli spaces in Appendix A. Moreover, we apply these method to the polynomials of degree n , and give some conjectures.

1 Quadratic rational maps

1.1 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\text{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\text{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\text{Rat}_2(\mathbf{C})$ by conjugation,

$$g \circ f \circ g^{-1} \in \text{Rat}_2(\mathbf{C}) \quad \text{for } g \in \text{PSL}_2(\mathbf{C}), f \in \text{Rat}_2(\mathbf{C}).$$

Two maps $f_1, f_2 \in \text{Rat}_2(\mathbf{C})$ are **holomorphically conjugate**, denoted by $f_1 \sim f_2$, if and only if there exists $g \in \text{PSL}_2(\mathbf{C})$ with $g \circ f_1 \circ g^{-1} = f_2$. The quotient space of $\text{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f .

Milnor introduced coordinates in $\mathcal{M}_2(\mathbf{C})$ as follows; for each $f \in \text{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i ; $\mu_i =$

$f'(z_i)$ ($1 \leq i \leq 3$). Consider the elementary symmetric functions of the three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad \sigma_3 = \mu_1\mu_2\mu_3.$$

These three multipliers determine f up to holomorphic conjugacy, and are subject only to the restriction that

$$\sigma_3 = \sigma_1 - 2.$$

Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to \mathbf{C}^2 with coordinates σ_1 and σ_2 (Lemma 3.1 in [Mil93]).

For each $\mu \in \mathbf{C}$ let $\text{Per}_n(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f which having a periodic point of period n and multiplier μ .

Each of $\text{Per}_1(\mu)$ and $\text{Per}_2(\mu)$ forms a straight lines as follows:

$$\begin{aligned} \text{Per}_1(\mu) &= \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1}) \right\} \\ \text{Per}_2(\mu) &= \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = -2\sigma_1 + \mu \right\}, \end{aligned}$$

(Lemmas 3.4 and 3.6 in [Mil93]).

Remark $\text{Per}_1(-1) \subseteq \text{Per}_2(1)$ by definition. But, in the case of $\mathcal{M}_2(\mathbf{C})$, it is clear that two families coincide.

1.2 Symmetry locus

By an automorphism of a quadratic rational map f , we will mean $g \in \text{PSL}_2(\mathbf{C})$ which commutes with f . The collection $\text{Aut}(f)$ of all automorphisms of f forms a finite group. It is clear that $\text{Aut}(\tilde{f})$ is isomorphic to $\text{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$.

The set

$$\mathcal{S} = \{ \langle f \rangle; \text{Aut}(f) \text{ is non-trivial} \} \subset \mathcal{M}_2(\mathbf{C})$$

is called the **symmetry locus**.

Corollary 1 *The symmetry locus \mathcal{S} of quadratic rational maps forms an irreducible algebraic curve as follows;*

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0. \quad (1)$$

Proof of Corollary 1.

$\text{Aut}(f)$ coincides with the group consisting of all permutations of the fixed points which preserve the multipliers. In the case of f has the three distinct fixed points, $\text{Aut}(f)$ has order 1, 2, or 6 according as three multipliers are distinct, two are equal, or all the three are equal, respectively, while, if f has multiple fixed points then $\text{Aut}(f)$ is non-trivial if and only if f has a triple fixed point. The multipliers μ_i are the roots of the equation:

$$\mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_1 + 2 = 0. \quad (2)$$

The equation (2) has multiple roots if and only if its discriminant is equal to zero. Hence we have

$$(\sigma_2 - 2\sigma_1 + 3)(2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36) = 0.$$

The first factor corresponds with $\text{Per}_1(1)$. Considering the line of the first factor ($\text{Per}_1(1)$) tangent to the curve of the second factor (S) with tangency of degree three, the second factor is the required equation. ■

The following result is obtained immediately by the definition of the envelope of the family of curves.

Corollary 2 *The envelope of $\{\text{Per}_1(\mu)\}_\mu$ coincides with the symmetry locus.*

Remark (Theorem 5.1. of [Mil93]) A quadratic rational map has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $f(z) = k(z + \frac{1}{z})$ with $k \in \mathbf{C} \setminus \{0\}$.

1.3 Real moduli space

Let $\text{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. Then the parameters σ_i ($1 \leq i \leq 3$) are all real, because the three fixed points and the corresponding multipliers are either all real or one real and a pair of complex conjugate numbers. According to J. Milnor, we define the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\text{Rat}_2(\mathbf{R})$ to be simply the real (σ_1, σ_2) -plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap \mathcal{M}_2(\mathbf{R})$, and denote by $\langle \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\text{Rat}_2(\mathbf{R})/\text{PGL}_2(\mathbf{R})) \setminus \{\langle a(x + \frac{1}{x}) \rangle_{\mathbf{R}}, \langle a(x - \frac{1}{x}) \rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle a(x \pm \frac{1}{x}) \rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

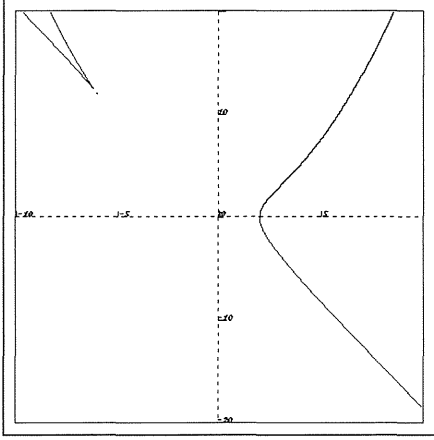


Figure 1: $\mathcal{M}_2(\mathbf{R})$ with the real cut of S .

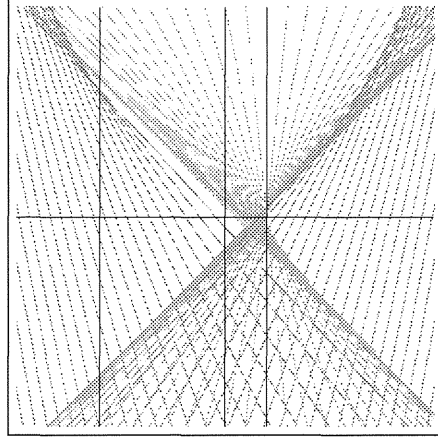


Figure 2: Lines $\{\text{Per}_1(\mu)\}$ in the real cut of the moduli space $\mathcal{M}_2(\mathbf{C})$

2 Cubic polynomials

2.1 Moduli space of cubic polynomials

Let $\text{Poly}_3(\mathbf{C})$ be the space of all cubic polynomials from \mathbf{C} to itself. The group $\text{Poly}_1(\mathbf{C})$ of affine transformations acts on the space $\text{Poly}_3(\mathbf{C})$ by conjugation,

$$g \circ p \circ g^{-1} \in \text{Poly}_3(\mathbf{C}) \quad \text{for} \quad g \in \text{Poly}_1(\mathbf{C}), \quad p \in \text{Poly}_3(\mathbf{C}).$$

Two maps $p_1, p_2 \in \text{Poly}_3(\mathbf{C})$ are **holomorphically conjugate**, denoted by $p_1 \sim p_2$, if and only if there exists $g \in \text{Poly}_1(\mathbf{C})$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\text{Poly}_3(\mathbf{C})$ under this action will be denoted by $\text{M}_3(\mathbf{C})$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of cubic polynomials p .

Doing the same as the case of quadratic rational maps, we introduce coordinates in $\text{M}_3(\mathbf{C})$ as follows; for each $p \in \text{Poly}_3(\mathbf{C})$, let $z_1, z_2, z_3, z_4 (= \infty)$ be the fixed points of p and μ_i the multipliers of z_i ; $\mu_i = p'(z_i)$ ($1 \leq i \leq 3$), and $\mu_4 = 0$. Consider the elementary symmetric functions of the four multipliers,

$$\begin{aligned} \sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\ \sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\ \sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\ \sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0. \end{aligned}$$

These multipliers determine uniquely p up to holomorphic conjugacy, and are subject only to the restriction that

$$3 - 2\sigma_1 + \sigma_2 = 0.$$

Hence the moduli space $M_3(\mathbb{C})$ is canonically isomorphic to \mathbb{C}^2 with coordinates σ_1 and σ_3 .

Proposition 1 *The locus $Per_1(\mu)$ forms a straight lines as follows:*

$$Per_1(\mu) = \left\{ \langle f \rangle \in M_3(\mathbb{C}); \sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu \right\}.$$

The locus $Per_2(\mu)$ forms an algebraic curve of degree three as follows:

$$\begin{aligned} Per_2(\mu) = \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbb{C}); \sigma_3^2 + (4\sigma_1^2 - (\mu + 57)\sigma_1 + 252)\sigma_3 - (4\mu - 16)\sigma_1^3 \right. \\ \left. + (61\mu - 252)\sigma_1^2 - (4\mu^2 + 246\mu - 1134)\sigma_1 - \mu^3 + 51\mu^2 \right. \\ \left. - 99\mu - 459 = 0 \right\}. \end{aligned}$$

Note that this curve is irreducible if and only if $\mu \neq 1$. In the case of $\mu = 1$,

$$Per_2(1) = Per_1(-1) \bigcup \left\{ \langle f \rangle \in \mathcal{M}_2(\mathbb{C}); \sigma_3 + 4\sigma_1^2 - 61\sigma_1 + 254 = 0 \right\}.$$

2.2 Symmetry locus

Using conjugation described in above, we can define symmetry locus of this moduli space as one in $\mathcal{M}_2(\mathbb{C})$, and we obtain next results.

Theorem 1 *The symmetry locus \mathcal{S} of cubic polynomials forms an irreducible algebraic curve:*

$$S(\sigma_1, \sigma_3) = 27\sigma_3 + (\sigma_1 - 6)(2\sigma_1 - 3)^2 = 0. \quad (3)$$

The following result is obtained immediately by the definition of the envelope of the family of curves.

Corollary 3 *The envelope of $\{Per_1(\mu)\}_\mu$ coincides with the symmetry locus.*

Remark A cubic polynomial has non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $p(z) = z^3 + az$.

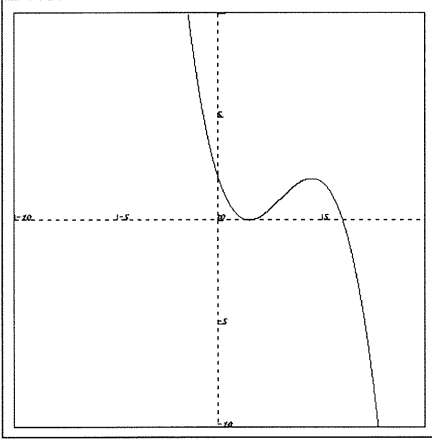


Figure 3: $M_3(\mathbf{R})$ with the real cut of \mathcal{S} .

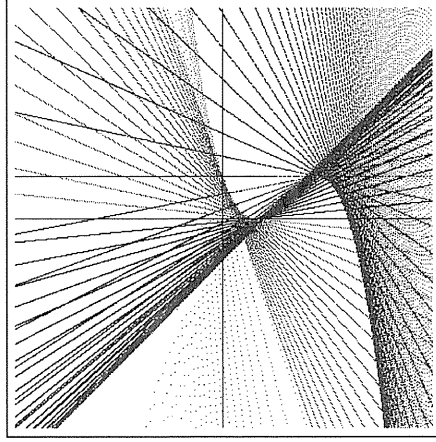


Figure 4: Lines $\{\text{Per}_1(\mu)\}$ in the real cut of the moduli space $M_3(\mathbf{C})$.

2.3 Real moduli space

Let $\text{Poly}_3(\mathbf{R})$ be the set of real cubic polynomials. By the same reason for the case of \mathcal{M}_2 , we define the real moduli space $M_3(\mathbf{R})$ for $\text{Poly}_3(\mathbf{R})$ to be simply the real (σ_1, σ_3) -plane. This notation needs some care when used: if we put $\mathcal{S}_{\mathbf{R}} = \mathcal{S} \cap M_3(\mathbf{R})$, and denote by $\langle \rangle_{\mathbf{R}}$ the real conjugacy class, then $(\text{Poly}_3(\mathbf{R})/\text{Poly}_1(\mathbf{R})) \setminus \{\langle x^3 + ax \rangle_{\mathbf{R}}, \langle -x^3 + ax \rangle_{\mathbf{R}}\}_{a \in \mathbf{R}^\times}$ is canonically isomorphic to $\mathbf{R}^2 \setminus \mathcal{S}_{\mathbf{R}}$, whereas there is a canonical two-to-one correspondence between $\{\langle \pm x^3 + ax \rangle\}_{a \in \mathbf{R}^\times}$ and $\mathcal{S}_{\mathbf{R}}$.

3 Polynomials of degree n

3.1 Moduli space of polynomials of degree n

Now we discuss about the moduli space $M_n(\mathbf{C})$ for the space, $\text{Poly}_n(\mathbf{C})$, of polynomials of degree n .

Doing the same as the case of cubic polynomials, we try introducing coordinates in $M_n(\mathbf{C})$ as follows; for each $p(z) \in \text{Poly}_n(\mathbf{C})$, let $z_1, \dots, z_n, z_{n+1}(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i ; $\mu_i = p'(z_i)$ ($1 \leq i \leq n$), and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the n

multipliers,

$$\begin{aligned}\sigma_{n,1} &= \mu_1 + \cdots + \mu_n, \\ \sigma_{n,2} &= \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \sum_{i=1}^{n-1} \mu_i \sum_{j>i}^n \mu_j, \\ &\dots \\ \sigma_{n,n} &= \mu_1\mu_2 \cdots \mu_n, \\ \sigma_{n,n+1} &= 0.\end{aligned}$$

Example 1 For example, we assume $p(z) \in \text{Poly}_4(\mathbb{C})$;

- fixed points: $z_1, z_2, z_3, z_4, \infty$
- multiplier: $\mu_1, \mu_2, \mu_3, \mu_4, 0$
- elementary symmetric functions:

$$\begin{cases} \sigma_{4,1} = \mu_1 + \mu_2 + \mu_3 + \mu_4 \\ \sigma_{4,2} = \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\ \sigma_{4,3} = \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 \\ \sigma_{4,4} = \mu_1\mu_2\mu_3\mu_4 \\ \sigma_{4,5} = 0 \end{cases}$$

Applying Fatou-index theorem to these fixed points;

$$\frac{1}{1-\mu_1} + \frac{1}{1-\mu_2} + \frac{1}{1-\mu_3} + \frac{1}{1-\mu_4} + \frac{1}{1-0} = 1, \quad (4)$$

where $\mu_i \neq 1$ ($1 < i < n$). Arranging this equation for the form of elementary symmetric functions;

$$4 - 3(\mu_1 + \mu_2 + \mu_3 + \mu_4) + 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4) - (\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4) = 0.$$

Hence we have

$$4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0. \quad (5)$$

For the equation (5), the cases $\mu_i = 1$ are also allowable.

Now we consider a polynomial $p(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0 \in \text{Poly}_4(\mathbb{C})$ that has at least two fixed points. After affine conjugation, we can assume they are 0 and 1. Then, we will solve the following question: “Do the four multipliers

$$\mu_0 = p'(0), \mu_1 = p'(1), \mu_2 = p'(z_2), \mu_3 = p'(z_3),$$

where z_1, z_2 are fixed points of $p(z)$, determine the five coefficients a_4, a_3, a_2, a_1, a_0 of $p(z)$?"

In fact, the following equations hold;

$$\begin{aligned} a_0 &= 0 && \text{because of } f(0) = 0, \\ a_1 &= \mu_0 && \text{because of } f'(0) = \mu_0, \\ a_2 &= a_4 + 3 - 2\mu_0 - \mu_1 && \text{because of } f'(1) = \mu_1, \\ a_3 &= 1 - a_4 - a_2 - \mu_0 && \text{because of } f(1) = 1, \end{aligned}$$

and a_4 is a common root of the following two equations;

$$\begin{aligned} A_1 &= (\mu_2^2 - 2\mu_3\mu_2 + \mu_3^2 - \mu_0^2 + 2\mu_1\mu_0 - \mu_1^2)a_4^4 + (-4\mu_0^3 + (4\mu_1 + 8)\mu_0^2 + (-4\mu_1^2 - 8)\mu_0 + 4\mu_1^3 - 8\mu_1^2 + 8\mu_1)a_4^3 + (-6\mu_0^4 + (-4\mu_1 + 28)\mu_0^3 + (4\mu_1^2 + 4\mu_1 - 44)\mu_0^2 + (-4\mu_1^3 + 4\mu_1^2 - 8\mu_1 + 32)\mu_0 - 6\mu_1^4 + 28\mu_1^3 - 44\mu_1^2 + 32\mu_1 - 16)a_4^2 + (-4\mu_0^5 + (-12\mu_1 + 32)\mu_0^4 + (-8\mu_1^2 + 64\mu_1 - 96)\mu_0^3 + (8\mu_1^3 - 96\mu_1 + 128)\mu_0^2 + (12\mu_1^4 - 64\mu_1^3 + 96\mu_1^2 - 64)\mu_0 + 4\mu_1^5 - 32\mu_1^4 + 96\mu_1^3 - 128\mu_1^2 + 64\mu_1)a_4 - \mu_0^6 + (-6\mu_1 + 12)\mu_0^5 + (-15\mu_1^2 + 60\mu_1 - 60)\mu_0^4 + (-20\mu_1^3 + 120\mu_1^2 - 240\mu_1 + 160)\mu_0^3 + (-15\mu_1^4 + 120\mu_1^3 - 360\mu_1^2 + 480\mu_1 - 240)\mu_0^2 + (-6\mu_1^5 + 60\mu_1^4 - 240\mu_1^3 + 480\mu_1^2 - 480\mu_1 + 192)\mu_0 - \mu_1^6 + 12\mu_1^5 - 60\mu_1^4 + 160\mu_1^3 - 240\mu_1^2 + 192\mu_1 - 64 = 0, \\ A_2 &= (\mu_2 + \mu_3 + \mu_0 + \mu_1 - 4)a_4^2 + (2\mu_0^2 - 4\mu_0 - 2\mu_1^2 + 4\mu_1)a_4 + \mu_0^3 + (3\mu_1 - 6)\mu_0^2 + (3\mu_1^2 - 12\mu_1 + 12)\mu_0 + \mu_1^3 - 6\mu_1^2 + 12\mu_1 - 8 = 0. \end{aligned}$$

Above two equations have common roots if and only if $\mu_0, \mu_1, \mu_2, \mu_3$ satisfy the equation (5). Since $\mu_0, \mu_1, \mu_2, \mu_3$ are the four multipliers of $p(z)$ and they should satisfy the equation (5), the two equations always have common roots. Hence five coefficients of $p(z)$ are calculated by its four multipliers, however, this calculation is not decisive when they have distinct two common roots.

For the case of $\text{Poly}_n(\mathbf{C})$, it is clear from (4) that the equation corresponds to (5) cannot have the term of $\sigma_{n,n}$. Hence we can put

$$c_0 + c_1\sigma_{n,1} + c_2\sigma_{n,2} + \cdots + c_{n-1}\sigma_{n,n-1} = 0$$

where c_k ($0 \leq k \leq n-1$) are functions of n variable.

Paying attention to the form of elementary symmetric functions, we obtain the following equation;

$$c_k = (-1)^k \frac{\binom{n-1}{k}^n}{\binom{n}{k}} = n - k.$$

where $\binom{n}{k}$ means binomial coefficient. For convenience, put $\sigma_{n,0} = 1$. we have

$$\sum_{k=0}^{n-1} (-1)^k (n-k) \sigma_{n,k} = 0. \quad (6)$$

Question Is the moduli space $M_n(\mathbb{C})$ for polynomials of degree n canonically isomorphic to \mathbb{C}^{n-1} with coordinates $\sigma_1, \sigma_2, \dots, \sigma_{n-2}$, and σ_n ?

3.2 Symmetry locus

Proposition 2 *A polynomial of degree four has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form*

$$\{z^4 + az\}, \quad a \in \mathbb{C}.$$

For a map $p(z)$ in this normal form, $\text{Aut}(p)$ is a cyclic group of order three.

Outline of proof. Let $p(z) \in \text{Poly}_4(\mathbb{C})$.

1. In the case of a map $p(z)$ with multiple fixed points.
 - (a) The case of $p(z)$ with a fixed point of order four: $\text{Aut}(p)$ is non-trivial.
 - (b) The case of $p(z)$ with a fixed point of order three: $\text{Aut}(p)$ is trivial.
 - (c) The case of $p(z)$ with two fixed points of order two: there is not such $p(z)$.
 - (d) The case of $p(z)$ with a fixed point of order two: $\text{Aut}(p)$ is trivial.
2. In the case of a map $p(z)$ with four distinct fixed points.
 - (a) The case of four distinct multipliers: $\text{Aut}(p)$ is trivial.
 - (b) The case that only two of multipliers are coincide: $\text{Aut}(p)$ is trivial.
 - (c) The case of two pair of same multipliers: there is not such $p(z)$.
 - (d) The case of three same multipliers: By an affine conjugation, if three fixed points (whose multipliers are same) are mapped on the vertices of a regular triangle whose barycenter is the origin and the other fixed point on the origin, then $\text{Aut}(p)$ is non-trivial. Otherwise $\text{Aut}(p)$ is trivial.

(e) The case of four same multipliers: there is not such $p(z)$.

Therefore a map $p(z)$ has non-trivial automorphisms if and only if $p(z)$ is in the case 1-(a) and the first part of 2-(d). We can check easily that these maps coincide with the normal form $\{z^4 + az\}$. ■

Conjecture A polynomial of degree n has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form

$$\left\{ z^n + \sum_{k|(n-1), k \neq n-1} A(k)z^k \right\}$$

where $A(k)$ are parameters in \mathbb{C} .

A Comparison between the quadratic rational maps and cubic polynomials

	Quadratic rational maps	Cubic polynomials
<i>Moduli Space</i>	$\mathcal{M}_2(\mathbb{C}) \simeq \mathbb{C}^2$	$\mathcal{M}_3(\mathbb{C}) \simeq \mathbb{C}^2$
<i>Real Moduli Space</i>	$\mathcal{M}_2(\mathbb{R}) \simeq \mathbb{R}^2$ <i>excepts on the symm. locus</i>	$\mathcal{M}_3(\mathbb{R}) \simeq \mathbb{R}^2$ <i>excepts on the symm. locus</i>
<i>Coordinates</i>	$(\sigma_1, \sigma_2), \quad \sigma_3 = \sigma_1 - 2$	$(\sigma_1, \sigma_3), \quad 3 - 2\sigma_1 + \sigma_2 = 0$
<i>Normal Forms</i>	<i>Fixed Pint Normal Form, etc.</i>	$\{f(z) = z^3 + az + b\}_{(a,b)}$
<i>Periodic Orbits</i>	$\text{Per}_1(\mu) :$ $\sigma_2 = (\mu + \frac{1}{\mu})\sigma_1 - (\mu^2 + \frac{2}{\mu})$ $\text{Per}_2(\mu) :$ $2\sigma_1 + \sigma_2 = \mu$ $\text{Per}_1(-1) = \text{Per}_2(1)$	$\text{Per}_1(\mu) :$ $\sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu$ $\text{Per}_2(\mu) :$ <i>cubic algebraic curve</i> $\text{Per}_1(-1) \subset \text{Per}_2(1)$
<i>Symmetry Locus</i>	<i>the envelope of $\{\text{Per}_1(\mu)\}$</i> <i>normal form : $\{k(z + \frac{1}{z})\}$</i>	<i>the envelope of $\{\text{Per}_1(\mu)\}$</i> <i>normal form : $\{z^3 + az\}$</i>
<i>Topological Partition</i>	<i>degree ± 2, monotone, unimodal, bimodal</i>	$\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$
<i>Hyp. Components</i>	B, C, D, E	A, B, C, D

A.1 Topological Partition

For map $f \in \mathcal{M}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree $+2$, else $f' < 0$ then the map of degree -2 .

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = \text{int}(f(S))$ contains no (resp. one, two) critical points [Mil93].

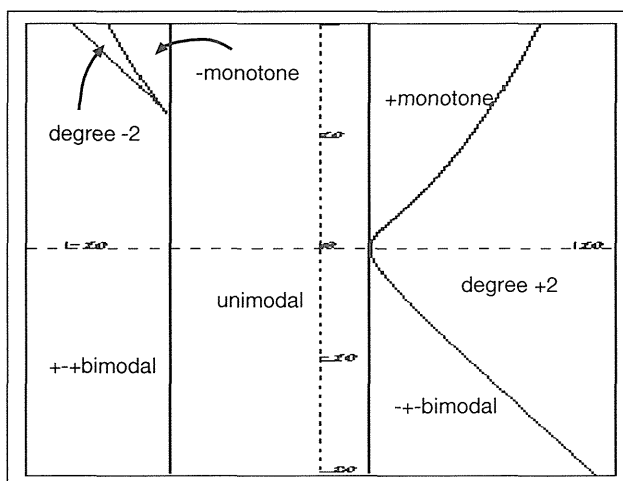


Figure 5: The topological partition of the $\mathcal{M}_2(\mathbf{R})$.

Boundary curves of Figure 5

$$CD_1 : \sigma_1 = 2$$

$$BC_1 : \sigma_1 = 6$$

$$\text{Symmetry locus} : S(\sigma_1, \sigma_2) = 0$$

where the curves CD_1 ($\text{Per}_1(0)$) and BC_1 are “center curve” defined in [NN93].

Remark Two curves BC_1 and CD_1 are boundary curves of the “unimodal” region.

For map $p \in M_3(\mathbf{R})$, if the real filled-in Julia set of p is a single point then it is said that p is in the class \mathcal{R}_0 . Let J be the smallest closed interval which contains the real filled-in Julia set of p . For $p \notin \mathcal{R}_0$, it is said that p belongs to the class \mathcal{R}_n if the graph of p intersected with $J \times J$ has n distinct components [Mil90].

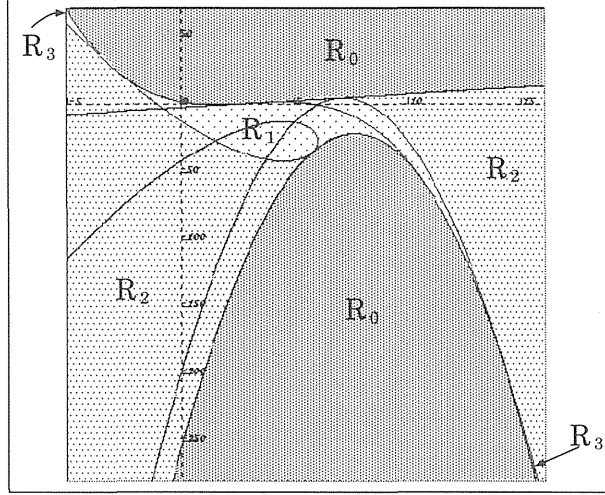


Figure 6: The topological partition of the $M_3(\mathbf{R})$.

Boundary curves of Figure 6

$$\text{Per}_1(1) : \sigma_1 - \sigma_3 - 2 = 0$$

$$\text{Preper}_{(1)}1 : -4\sigma_1^2 + 57\sigma_1 - \sigma_3 - 198 = 0$$

$$\text{Symmetry locus} : S(\sigma_1, \sigma_3) = 0$$

$$\text{Per}_2(1) : -8\sigma_1^3 + 180\sigma_1^2 - 1809\sigma_1 - 27\sigma_3 + 6966 = 0$$

$$\begin{aligned} \text{Preper}_{(1)}2 : & 64\sigma_1^6 - 1152\sigma_1^5 + 7776\sigma_1^4 + (432\sigma_3 - 25056)\sigma_1^3 \\ & + (-3888\sigma_3 + 41796)\sigma_1^2 + (8748\sigma_3 - 34992)\sigma_1 + 729\sigma_3^2 \\ & - 45198\sigma_3 + 543105 = 0 \end{aligned}$$

A.2 Hyperbolic components

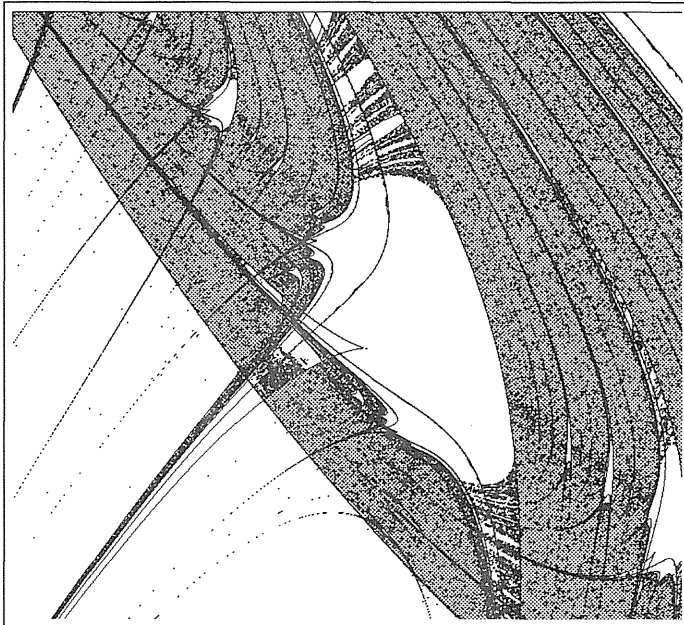


Figure 7: “Swallow configuration” in the moduli space $\mathbf{M}_3(\mathbf{C})$ of the cubic polynomials. $0.8 < \sigma_1 < 3.7$, $-37.6 < \sigma_3 < -20$.

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components.

The hyperbolic components of $\mathcal{M}_2(\mathbf{C})$ can be divided into the following four classes; Bitransitive-, Capture-, Disjoint-, and Escape-types [Ree90].

The hyperbolic components of $\mathbf{M}_3(\mathbf{C})$ can be divided into the following four classes; Adjacent-, Bitransitive-, Capture-, and Disjoint-types [Mil90].

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